Allotted time : 2h

Exercice 1 We consider the function from $\mathbb{R}^3$ into $\mathbb{R}$

$$f(x) = \frac{1}{2}(x_2 - x_1)^2 + \frac{1}{2}(x_1 + 2)^2 + \frac{1}{2}x_3^2.$$

1. Compute the gradient $\nabla f(x)$ and the Hessian $Hf(x)$.
2. Show that f has a unique global minimum at $x^\star$.
3. Compute $x^\star$ and $f(x^\star)$.
4. Let $\hat{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Compute $\nabla f(\hat{x})$.
5. Compute the minimum of the function $h_1 : \mathbb{R}^+ \to \mathbb{R}$

$$h_1(\alpha) = f(\hat{x} - \alpha \nabla f(\hat{x})).$$

6. Show that $\hat{d} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ is a descent direction at $\hat{x}$.
7. Show that it is possible to join $x^\star$ starting from $\hat{x}$ in direction $\hat{d}$ in one step.

Exercice 2 Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = (x_1 + 1)^2 + (x_2 - 1)^2$ and the minimization problem

$$\text{Find } \inf f(x) \text{ under constraint } x_1 x_2 = 1.$$

1. Draw a sketch to illustrate the problem
   (a) Draw the admissible domain D_a
   (b) Describe the level curves $f^{-1}(c)$ for $c \in \{0, 1, 4\}$.
   (c) Add these level curves on the previous graph
2. Does the problem have a solution (justify your answer) ?
3. Show that the Lagrange Theorem hypotheses are satisfied.
4. Write the Lagrangian of the problem and compute its gradient with respect to x.
5. Compute the solution using the Lagrange theorem. It may be useful to use the identity $x^2 \pm x = (x \pm \frac{1}{2})^2 - \frac{1}{4}$.
6. Add the points $x^\star$ such that $f(x^\star) = \inf_{x \in D_a} f(x)$ on the graph of question 1.
7. *Bonus.* We change the constraint into $x_1 x_2 \leq 1$. Is the solution modified ? Justify your answer. Same question for the opposite constraint $x_1 x_2 \geq 1$.