AIMS Sénégal - March 2022

Numerical Optimization catching up exam   Allotted time : 2h

The students are allowed to have a one side A4 sheet of course notes.
This sheet must be returned with the assigment

Exercice 1 We consider the function from $\mathbb{R}^2$ into $\mathbb{R}$

$$f(x) = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - x_1)^2.$$

1. Compute the gradient $\nabla f(x)$ and the Hessian $Hf(x)$.
2. Show that f has a unique global minimum at $x^\star$. Compute $x^\star$ and $f(x^\star)$.
3. Let $\hat{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Compute $\nabla f(\hat{x})$. Show that $\hat{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a descent direction at $\hat{x}$.
4. Compute the minimum of the function $h_1 : \mathbb{R}^+ \to \mathbb{R}$
$$h_1(\alpha) = f(\hat{x} + \alpha\hat{d}).$$
5. Compute $d^\star = -Hf(\hat{x})^{-1}\nabla f(\hat{x})$. What can you say about this direction ? Is it possible to join $x^\star$ starting from $\hat{x}$ in one step ?

Exercice 2 Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1 - x_2$ and the minimization problem

Find inf $f(x)$ under constraint $x_1^2 + x_2^2 = 4$.

1. Draw a sketch to illustrate the problem : draw the admissible domain D_a and the level curves $f^{-1}(c)$ for $c \in \{-1, 0, 1\}$.
2. Does the problem have a solution (justify your answer) ?
3. Show that the Local Extrema Theorem hypotheses are satisfied.
4. Compute the solution using this theorem. Add the point $x^\star$ on your drawing.

Exercice 3 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 + (x_2 - 1)^2$ and the minimization problem :

Find inf $f(x)$ under constraints $x_1 + x_2 \leq 0$ and $x_1 - x_2 \leq 0$.

1. Draw a sketch to illustrate the problem : draw the admissible domain D_a and the level curves $f^{-1}(c)$ for $c \in \{0, 1, 4\}$.
2. Does the problem have a solution (justify your answer) ?
3. Show that the hypotheses of the Karush-Kuhn-Tucker theorem are satisfied.
4. Find the points and Lagrange multipliers satisfying the KKT conditions.
5. Give the value of $p^\star = \inf_{D_A} f(x)$. Where is it reached ? Which constraints are active at $x^\star$? Add this point on your drawing.