AIMS Sénégal

Numerical Optimization final exam - 2022/02/18 Allotted time : 2h

The students are allowed to have a one side A4 sheet of course notes.
This sheet must be returned with the assigment

Exercice 1 We consider the function from $\mathbb{R}^2$ into $\mathbb{R}$

$$f(x) = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(x_1 - 2)^2 + \frac{1}{2}(x_2 - 1)^2.$$

1. Compute the gradient $\nabla f(x)$ and the Hessian $Hf(x)$.
2. Show that f has a unique global minimum at $x^\star$. Compute $x^\star$ and $f(x^\star)$.
3. Let $\hat{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Compute $\nabla f(\hat{x})$. Show that $d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a descent direction at $\hat{x}$.
4. Compute the minimum of the function $h_1 : \mathbb{R}^+ \to \mathbb{R}$
$$h_1(\alpha) = f(\hat{x} + \alpha d).$$
5. Let $\hat{d} = -\nabla f(\hat{x})$. Compute the minimum of $h_2 : \mathbb{R}^+ \to \mathbb{R}$
$$h_2(\alpha) = f(\hat{x} + \alpha \hat{d}).$$
6. Deduce from the previous question that $\hat{d}$ is the "best" descent direction at $\hat{x}$.

Exercice 2 Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1 + x_2$ and the minimization problem

Find inf $f(x)$ under constraint $x_1^2 + x_2^2 = 2$.

1. Draw a sketch to illustrate the problem : draw the admissible domain D_a and the level curves $f^{-1}(c)$ for $c \in \{-1, 0, 1\}$.
2. Does the problem have a solution (justify your answer) ?
3. Show that the Local Extrema Theorem hypotheses are satisfied.
4. Compute the solution using this theorem.

Exercice 3 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1 x_2$ and the minimization problem :

Find inf $f(x)$ under constraints $x_1^2 + x_2^2 \leq 2$ and $x_1 + x_2 \geq 0$.

1. Draw a sketch to illustrate the problem : draw the admissible domain D_a and the level curves $f^{-1}(c)$ for $c \in \{-1, 0, 1\}$.
2. Does the problem have a solution (justify your answer) ?
3. Show that the hypotheses of the Karush-Kuhn-Tucker theorem are satisfied.
4. Find the points and Lagrange multipliers satisfying the KKT conditions.
5. Give the value of $p^\star = \inf_{D_A} f(x)$. Where is it reached ?