$$f(x) = \frac{1}{2}(x_1+x_2)^2 + \frac{1}{2}(x_1-2)^2 + \frac{1}{2}(x_2-1)^2$$

1) Compute $\nabla f(x)$ and $Hf(x)$

2) Show that f has a ! global minimum $x^* = \operatorname{argmin} f(x)$

write $f(x) = \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + c$

$$f(x) = \frac{1}{2}\langle A_0(x-x_0), x-x_0\rangle + c_0$$

1) $\nabla f(x) = \begin{pmatrix} 2x_1 + x_2 - 2 \\ x_1 + 2x_2 - 1 \end{pmatrix}$ $Hf(x) = \begin{pmatrix} 2 & +1 \\ 1 & 2 \end{pmatrix}$

$\lambda = \{1, 3\} > 0$ $Hf(x) \in S^2_{++}$

1st possibility: f is strictly convex and $\nabla f(x) = 0$ has a ! solution because

$\nabla f(x) = \quad Hf(x)x - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad Hf(x) \in S^2_{++}$

$\Rightarrow$! global minimum.

2nd possibility $f(x) = \frac{1}{2}\langle Hf(0)x, x\rangle + \langle \nabla f(0), x\rangle + f(0)$

$Hf(0) \in S^2_{++} \Rightarrow$! global min.

$$\nabla f(x) = \begin{pmatrix} 2x_1 + x_2 - 2 \\ x_1 + 2x_2 - 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = Hf(0)x - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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$$\nabla f(x)=0 \iff Hf(0)x - \begin{pmatrix}2\\1\end{pmatrix}=0$$

linear system $\quad Ax=b$

$$A=Hf(0) \qquad b=\begin{pmatrix}2\\1\end{pmatrix}$$

$A\in S^2_{++} \Rightarrow$ invertible

$$\exists! \quad Ax^*=b$$

1) find x^*
2) write Taylor expansion at x^* instead of 0

$$f(x) = \frac{1}{2}(x_1+x_2)^2 + \frac{1}{2}(x_1-2)^2 + \frac{1}{2}(x_2-1)^2$$

$$= \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + c \qquad \text{here } x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = Hf(0)$$

$$b = \nabla f(0) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$c = \frac{5}{2} = f(0)$$

$$= \frac{1}{2}\langle \bar{A}(x-x_0), x-x_0\rangle + c_0$$

$$\bar{A},\ x_0,\ c_0 \ ?$$

$$f(x) = \frac{1}{2}\langle Hf(x_0)(x-x_0), x-x_0\rangle + \langle \nabla f(x_0), (x-x_0)\rangle + f(x_0)$$

choose $x_0 = x^*$

x^* solution of $\nabla f(x) = 0$ $\quad x^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$f(x) = \frac{1}{2} \langle A(x - x^*), x - x^* \rangle + f(x^*)$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad f(x^*) = \frac{3}{2}$$

$$f(x) = \frac{1}{2} \langle A \left(x - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), x - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle + \frac{3}{2}$$

$$f(x) = \frac{1}{2} \left(2(x_1 - 1)^2 + 2x_2^2 + 2(x_1 - 1)x_2 \right) + \frac{3}{2}$$

$$= (x_1 - 1)^2 + x_2^2 + x_2(x_1 - 1) + \frac{3}{2}$$

$$= \frac{1}{2} \langle A(x - x^*), x - x^* \rangle + \frac{3}{2}$$

the level curves of f : $f^{-1}(c)$ are

$$\left\{ \frac{1}{2} \langle A(x - x^*), (x - x^*) \rangle + \frac{3}{2} = c \right\}$$

if $c < \frac{3}{2}$ $\quad f^{-1}(c) = \emptyset$

if $c = \frac{3}{2}$ $\quad f^{-1}(c) = \{x^*\}$

if $c > \frac{3}{2}$ $\quad f^{-1}(c)$ ellipse

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ex1 $f(x) = \frac{1}{2}(x_1 - 2)^2 + \frac{1}{2}(x_2 - 3)^2$

$= \frac{1}{2} \langle Id(x - x_0), x - x_0 \rangle$

with $x_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$f^{-1}(0) = \{x_0\}$

$f^{-1}(1) =$ circle centered at x_0 radius $\sqrt{2}$

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ex2 $f(x) = 2x_1^2 + 3(x_2 - 1)^2$

$= \frac{1}{2} \langle A(x - x_0), x - x_0 \rangle$

$A = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$ $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $f^{-1}(0) = \{x_0\}$

$f^{-1}(c) = \emptyset$ for $c < 0$

$f^{-1}(c)$ ellipse centered at x_0 for $c > 0$

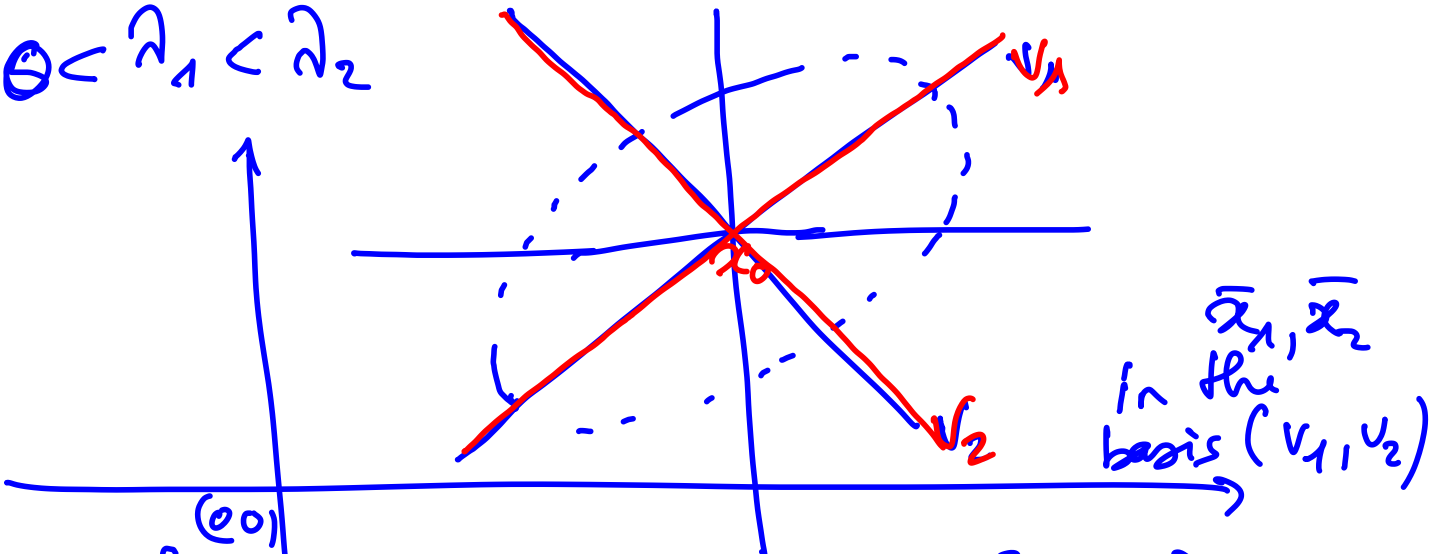
$f^{-1}(c) : \{x \in \mathbb{R}, f(x) = c\}$ level curve of level c.

$$f(x) = \frac{1}{2}\langle A(x-x_0), x-x_0 \rangle$$

change of basis for $A = MDM^{-1}$

with $M = (v_1, v_2)$ $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$0 < \lambda_1 < \lambda_2$



$$f^{-1}(c) = \left\{ \left(\frac{\bar{x}_1 - \bar{x}_{0_1}}{\lambda_1}\right)^2 + \left(\frac{\bar{x}_2 - \bar{x}_{0_2}}{\lambda_2}\right)^2 = c \right\}$$

3) $\hat{x} = \begin{pmatrix}0\\1\end{pmatrix}$; $\nabla f(\hat{x}) = \begin{pmatrix}-1\\1\end{pmatrix}$ Show that $d = \begin{pmatrix}1\\0\end{pmatrix}$ is a descent direction $\left\langle \begin{pmatrix}-1\\+1\end{pmatrix}, d \right\rangle = -1 < 0$

4) $h_1(\alpha) = f(\hat{x} + \alpha d) = f\begin{pmatrix}0+\alpha\\1+0\end{pmatrix} = f\begin{pmatrix}\alpha\\1\end{pmatrix}$

$= \frac{1}{2}(\alpha+1)^2 + \frac{1}{2}(\alpha-2)^2 + \frac{1}{2}(1-1)^2$

$= \frac{1}{2}(\alpha^2 + 2\alpha + 1 + \alpha^2 - 4\alpha + 4) = \alpha^2 - \alpha + \frac{5}{2}$

Compute the minimum: $h_1'(\alpha) = 2\alpha - 1$

$\alpha_{opt} = \frac{1}{2}$ $\quad h_1(\frac{1}{2}) = \frac{9}{4}$

5) Same question with $\hat{d} = -\nabla f(\hat{x})$

$\hat{d} = -\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$ $\quad h_2(\alpha) = f\left(\begin{pmatrix}0\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\end{pmatrix}\right) = f\begin{pmatrix}\alpha\\1-\alpha\end{pmatrix}$

$h_2(\alpha) = \frac{1}{2}(2\alpha^2 - 4\alpha + 5)$ $\quad h_2'(\alpha) = 2\alpha - 2$

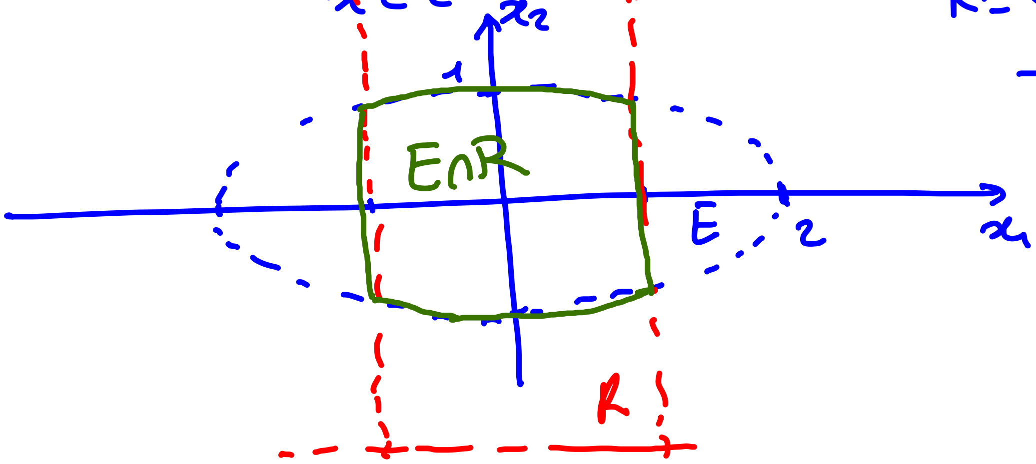
$\alpha_{opt} = 1$ $\quad h_2(1) = \frac{3}{2}$

$E = \left\{ x \in \mathbb{R}^2 \;\; \left(\frac{x_1}{b}\right)^2 + \left(\frac{x_2}{a}\right)^2 \le 1 \right\} = \left\{ \frac{x_1^2}{4} + x_2^2 \le 1 \right\}$

$R = \left\{ x \in \mathbb{R}^2 \quad |x_1| \le a \quad |x_2| \le b \right\}$

1) sketch $C = E \cap R$ for $a = 1$ $b = 2$, show C convex

2) study $\inf\limits_{x \in C} f(x)$ $\qquad f(x) = 2x_1^2 + x_2^2 + 5x_1x_2$



$R = \left\{ \begin{array}{l} -1 \le x_1 \le 1 \\ -2 \le x_2 \le 2 \end{array} \right\}$

C = intersection of 2 convex therefore convex.

$f(x) = 2x_1^2 + x_2^2 + 5x_1x_2$

1) does the Pb $\inf_{x \in C} f(x)$ has a solution?

yes because f continuous and C closed bounded $\Leftrightarrow$ compact

2) Apply KKT theorem

$C : \mathbb{R}^2 \longrightarrow \mathbb{R}^5$

$$x \longmapsto \begin{pmatrix} \frac{x_1^2}{4} + x_2^2 - 1 \\ x_1 - 1 \\ -1 - x_1 \\ x_2 - 2 \\ -2 - x_2 \end{pmatrix}$$

$$JC(x) = \begin{pmatrix} \frac{x_1}{2} & 2x_2 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

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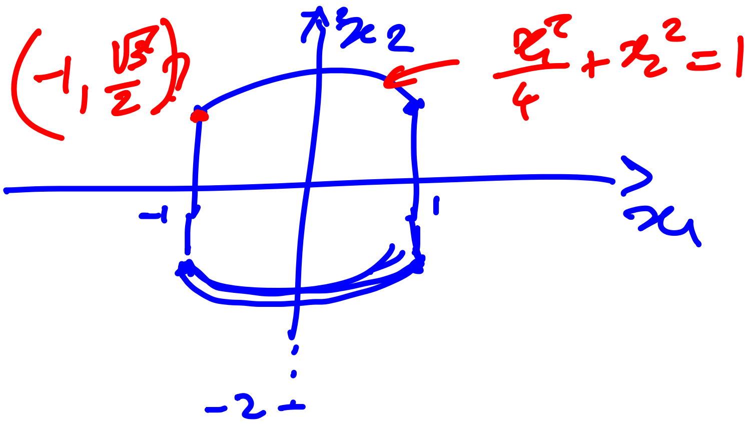
$$JC = \begin{pmatrix} \frac{x_1}{2} & 2x_2 \\ 1 & 0 \\ -1 & 0 \\ \cancel{0} & \cancel{1} \\ \cancel{0} & \cancel{-1} \end{pmatrix}$$



$$f(x) = 2x_1^2 + x_2^2 + 5x_1x_2$$
$$= 2 + \frac{3}{4} - \frac{5\sqrt{3}}{2} < 0 = f(0,0)$$

$$JC(x^*) = \begin{pmatrix} \frac{x_1}{2} & 2x_2 \end{pmatrix} \qquad Rk = 1$$

$$\text{or} \quad \begin{pmatrix} \frac{x_1}{2} & 2x_2 \\ -1 & 0 \end{pmatrix} \qquad Rk = 2$$

$$\text{or} \quad \begin{pmatrix} \frac{x_1}{2} & 2x_2 \\ 1 & 0 \end{pmatrix} \qquad Rk = 2$$