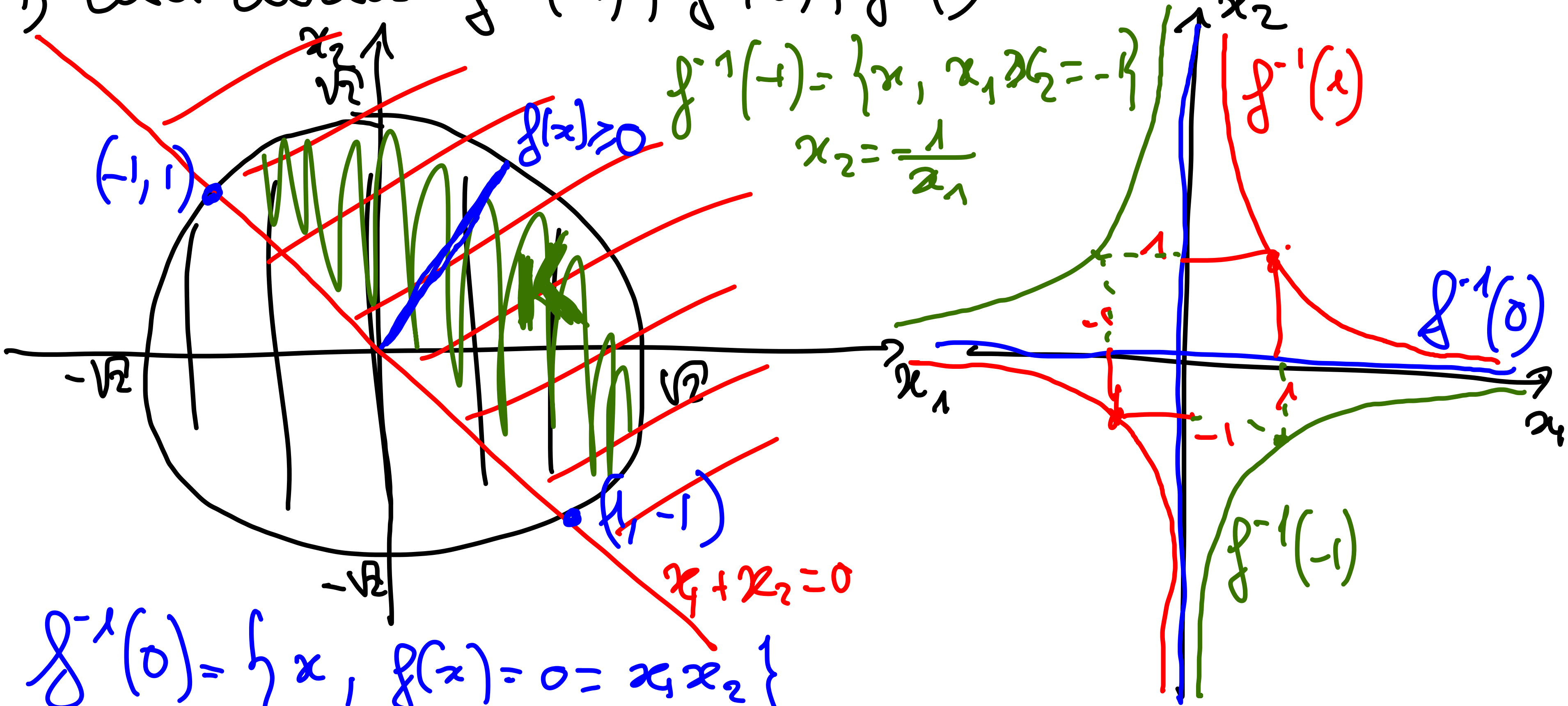
14/02/2024 exam 2022

Exercise 3) $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x) = x_1 x_2$

$K = \{ x_1^2 + x_2^2 \leq 2 \ , \ x_1 + x_2 \geq 0 \} = \{ x_1^2 + x_2^2 \leq 2 \} \cap \{ x_1 + x_2 \geq 0 \}$

1) Level curves $f^{-1}(-1)$, $f^{-1}(0)$, $f^{-1}(1)$ and K



$f^{-1}(0) = \{ x, \ f(x) = 0 = x_1 x_2 \}$

2) existence: f continuous on bounded set K has a minimum and a maximum.

3) KKT qualification of constraints

$c_1(x) = x_1^2 + x_2^2 - 2 \qquad c_2(x) = -x_1 - x_2$

$JC(x) = \begin{pmatrix} 2x_1 & 2x_2 \\ -1 & -1 \end{pmatrix}$ * $\quad Rk\, JC(x) = 2$ if $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \lambda \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \lambda \in \mathbb{R}$

$f\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 < f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq x^* \qquad x_1^* = x_2^*$

$2x_1^2 \leq 2 \quad x_1^2 \leq 1 \quad -x_1 - x_1 \leq 0 \qquad x_1 \geq 0$

x^* is not such that $x_1^* = x_2^*$ $\quad f(x_1, x_1) \geq 0 > f\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$Rk\, JC(x^*) = 2$

Apply the KKT Theorem

stars everywhere $z \in \mathbb{R}^2$ at the solution x^*, z^*

$$l(x, z) = f(x) + \langle z, c(x) \rangle$$
$$= x_1 x_2 + z_1(x_1^2 + x_2^2 - 2) + z_2(-x_1 - x_2)$$

$$\nabla_x l(x, z) = \begin{pmatrix} x_2 + 2z_1 x_1 - z_2 \\ x_1 + 2z_1 x_2 - z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1^2 + x_2^2 - 2 \leq 0$

$-x_1 - x_2 \leq 0$

$z_1 \geq 0$

$z_2 \geq 0$

$$\left.\begin{array}{l} z_1(x_1^2 + x_2^2 - 2) = 0 \\ z_2(x_1 + x_2) = 0 \end{array}\right\}$$

Complementary Relaxation

$z_i c_i^I(x) = 0$

$i = 1, \dots, p$

	$z_1 = 0$	$x_1^2 + x_2^2 = 2$
$z_2 = 0$	$x^* = 0, 0$ NO	x^* ? $(-1, 1)$ $(1, -1)$
$x_1 + x_2 = 0$	$x^* = 0$ NO	? Nothing new $x^* = (-1, 1), (1, -1)$

$p^* = -1 = f(x^*)$

1) Suppose $z_1 = z_2 = 0 \Rightarrow \nabla_x \ell(x, z) = 0 \Rightarrow x = (0, 0)$

but $f\begin{pmatrix}-1\\-1\end{pmatrix} < f\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x^* \neq (0,0)$

2) Suppose $z_1 = 0$ and $x_1 + x_2 = 0$

$x_1 = x_2$ and $x_1 = -x_2 \Rightarrow x^* = (0,0)$ again

3) Suppose $z_2 = 0$ and $x_1^2 + x_2^2 = 2$

$x_2 = -2z_1 x_1$ and $x_1 = -2z_1 x_2$

$= 4z_1^2 x_2 \Rightarrow$ either $x_2 = 0$ or $z_1^2 = \frac{1}{4}$

- $x_1^2 = 2$ $(\sqrt{2}, 0)$ or $(-\sqrt{2}, 0)$ $f(x^*) = 0$ NO
- $z_1 = \frac{1}{2}$ $x_1 = -x_2 = \pm 1$

answer questions: go back to proof of KKT with $C_j^I(x) \leq 0$ replaced by $C_j^I(x) + t_j^2 = 0$

why $z^* \geq 0$ and $z_i C_i^I(x) = 0$

Cas $z_2 = 0$

$$\left.\begin{array}{l} x_2 + 2z_1x_1 = 0 \\ x_1 + 2z_1x_2 = 0 \end{array}\right| \Rightarrow x_2 - 4z_1^2x_2 = 0$$

$$x_2(1 - 4z_1^2) = 0$$

$$x_1 = -2z_1x_2$$

1st cas : $x_2 = 0$ NO ! $\quad f(x_1, x_2) = 0 > f(-1, 1)$

2nd cas $\quad 1 - 4z_1^2 = 0 \quad z_1^2 = \frac{1}{4} \Rightarrow z_1 = \frac{1}{2}$

$$x_1 = -2z_1x_2 = -x_2 \qquad x_1^2 + x_1^2 = 2$$

$$x_1^2 = 1 \qquad x_1 = \pm 1 \qquad x_2 = \mp 1$$

$$x^* = (1, -1) \text{ or } (-1, 1) \qquad f(x^*) = -1$$