$f(x) = 2x_1^2 + 3x_2^2 + 2x_1x_2$ inf $f(x)$

$C_1(x) = x_1^2 + 4x_2^2 - 1$ $c(x) \leq 0$

$C_2(x) = 1 - x_1 - x_2$

Lagrangian $\ell(x, z) = f(x) + \langle z, c(x) \rangle$

$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ $\ell(x, z) = f(x) + z_1(x_1^2 + 4x_2^2 - 1) + z_2(1 - x_1 - x_2)$

qualification $JC(x) = \begin{pmatrix} 2x_1 & 8x_2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} \nabla C_1^T \\ \nabla C_2^T \end{pmatrix}$

if both constraints are active at x^* $Rk\, JC(x^*) = 2$

$2x_1 = -8x_2$ $x_1 = -4x_2$ $0 = 1 + 4x_2 - x_2$ $x_2 = -\frac{1}{3}$

absurde $\Rightarrow Rk = 2$

if c_1 active only $Rk\ 1$ because $(0,0) \neq x^*$

if c_2 active only $Rk\ 1$ because $(1,1) \neq (0,0)$

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1) $\nabla_x \ell(x,z) = \begin{pmatrix} 4x_1 + 2x_2 \\ 2x_1 + 6x_2 \end{pmatrix} + 2z_1 \begin{pmatrix} x_1 \\ 4x_2 \end{pmatrix} + z_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0$

2) $x_1^2 + 4x_2^2 - 1 \leq 0$
$1 - x_1 - x_2 \leq 0$

3) $z_1 \geq 0 \quad z_2 \geq 0$

4) $z_1(x_1^2 + 4x_2^2 - 1) = 0$
$z_2(1 - x_1 - x_2) = 0$

	$z_1 = 0$	$x_1^2 + 4x_2^2 = 1$
z_2		
$x_1 + x_2 = 1$	good case for min	

$$\begin{cases} 4x_1 + 2x_2 - z_2 = 0 \\ 2x_1 + 6x_2 - z_2 = 0 \\ 1 - x_1 - x_2 = 0 \end{cases}$$

$2x_1 = 4x_2 \quad x_1 = 2x_2$

$x_2 = \frac{1}{3} \quad x_1 = \frac{2}{3}$

$\boxed{\text{ellipse} \iff \frac{1}{2}\langle Ax, x\rangle = \text{cte}}$ #

$$f(x) = ax^2 + by^2 = \frac{1}{2}\langle Ax, x\rangle$$

1) $$= \frac{1}{2}\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

$$\nabla f(\cancel{x}) = \begin{pmatrix} 2ax \\ 2by \end{pmatrix} \qquad Hf(x) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

$$f(X) = f(0) + \langle \nabla f(0), X\rangle + \frac{1}{2}\langle Hf(0)X, X\rangle$$

$$A = Hf(x) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

2) f strictly convex $\iff Hf \in S^n_{++}$

$\iff a, b > 0$

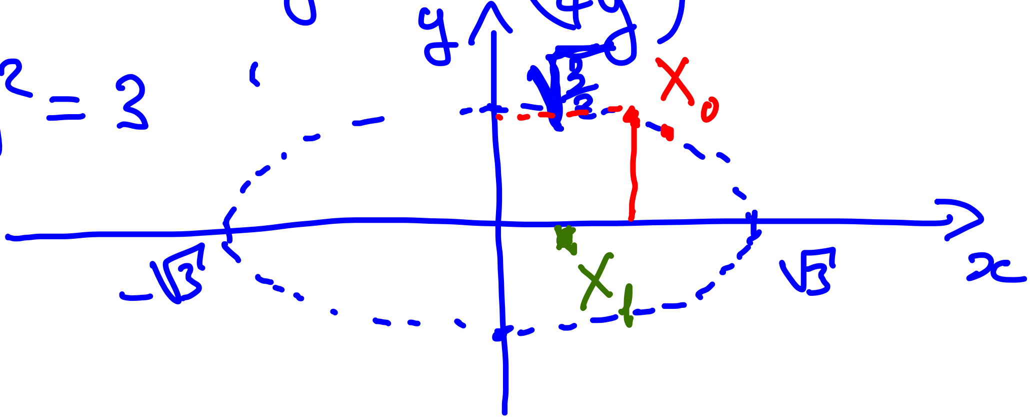
3) if $\cancel{x}^* \exists$, because of f strictly convex x^* is global and unique

$\nabla f(\cancel{x}^*) = 0$ has a solution $x^* = 0_{\mathbb{R}^2}$

for instance $f(x) = e^x$ is strictly convex doesn't have a global minimum

4) $a = 1$, $b = 2$ $\qquad \nabla f(\cancel{x}) = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$

5) $x^2 + 2y^2 = 3$



6) $X_0 = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad f(X_0) = 1^2 + 2\cdot 1^2 = 3$

$X_0 \in f^{-1}(3)$

7) $\nabla f(X_0) = \begin{pmatrix}2\\4\end{pmatrix} = g_0$

8) $X_1 = X_0 - \alpha \nabla f(X_0) = \begin{pmatrix}1\\1\end{pmatrix} - \alpha\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}1-2\alpha\\1-4\alpha\end{pmatrix}$

$h_0(\alpha) = f\left(X_0 - \alpha \nabla f(X_0)\right)$

$= (1-2\alpha)^2 + 2(1-4\alpha)^2 = 1 - 4\alpha + 4\alpha^2 + 2 - 16\alpha + 32\alpha^2$

$= 36\alpha^2 - 20\alpha + 3$

9) $h_0'(\alpha) = 72\alpha - 20$: $h'(\alpha_{opt}) = 0$

$\alpha_0 = \frac{20}{72} = \frac{5}{18}$ $\quad A = \begin{pmatrix}2 & 0\\0 & 4\end{pmatrix}$

10) $\alpha_k = \dfrac{-\langle g_k, d_k\rangle}{\langle A d_k, d_k\rangle}$

$\alpha_0 = \dfrac{+\langle g_0, g_0\rangle}{\langle A g_0, g_0\rangle} = \dfrac{\langle \begin{pmatrix}2\\4\end{pmatrix}\cdot\begin{pmatrix}2\\4\end{pmatrix}\rangle}{\langle A\begin{pmatrix}2\\4\end{pmatrix}\cdot\begin{pmatrix}2\\4\end{pmatrix}\rangle}$

$= \dfrac{4+16}{\langle \begin{pmatrix}4\\16\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix}\rangle} = \dfrac{20}{8+64} = \dfrac{20}{72} = \dfrac{5}{18}$

$f(x) \in P^2(\mathbb{R})$

$f(x) = ax^2 + bx + c$ $\quad f''(x) = 2a$

$f(x) = f(0) + f'(0)x + o(|x|)$ TF order 1

$= c + bx + \underbrace{o(|x|)}_{ax^2}$

$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + 0$ because $f^{(n)}(x) = 0$ for $n \geq 3$

$= c + bx + \frac{1}{2}\, 2a\, x^2$

$\Delta = b^2 - 4ac \quad \left(\Delta' = \left(\frac{b}{2}\right)^2 - ac \quad x = \frac{-b/2 \pm \sqrt{\Delta'}}{a}\right)$

$f(x) = 2x_1^2 + 3x_2^2 + 2x_1x_2$

$f(\vec{0}) = 0$

$\nabla f(x) = \begin{pmatrix} 4x_1 + 2x_2 \\ 6x_2 + 2x_1 \end{pmatrix} \qquad \nabla f(\vec{0}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$Hf(x) = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} = Hf(\vec{0}) = A$

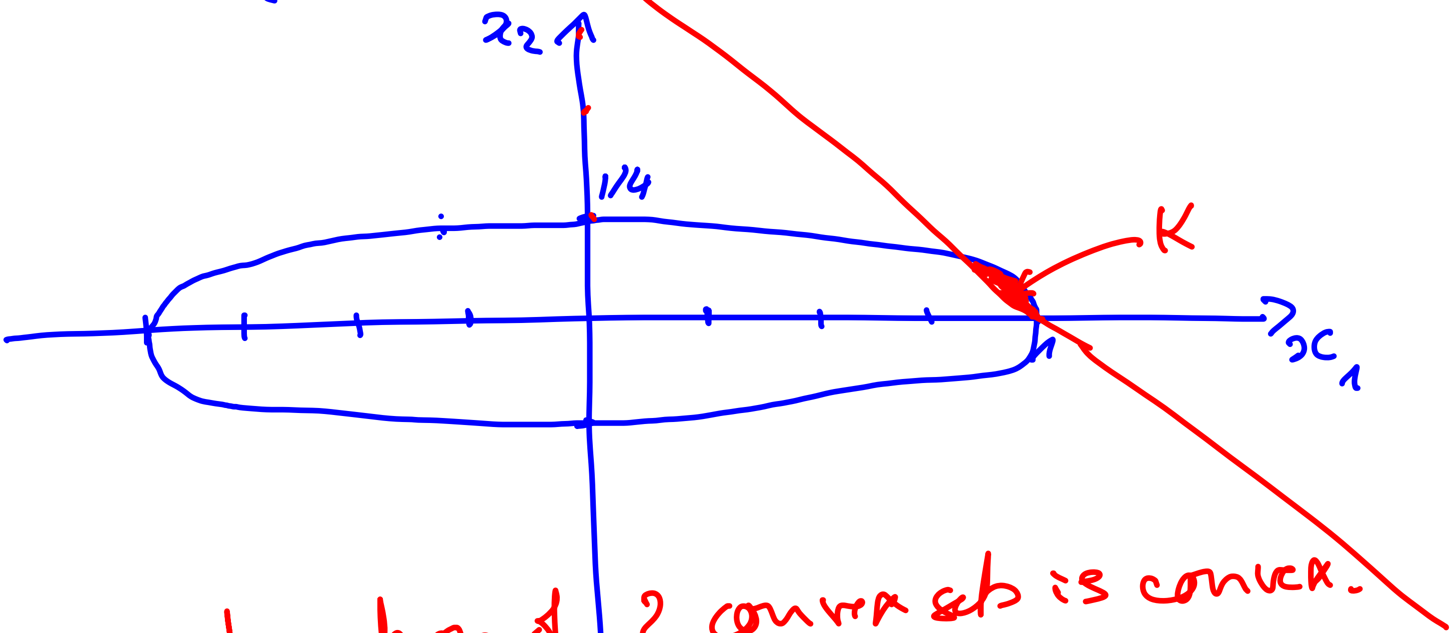
$f(x) = \frac{1}{2} \langle Ax, x \rangle$

$\det(A - \lambda I) = (4-\lambda)(6-\lambda) - 4 = \lambda^2 - 10\lambda + 20$

$\Delta = 25 - 20 = 5 \qquad \lambda = 10 \pm \sqrt{5} > 0$

$A \in S^2_{++} \quad \Rightarrow f$ has a unique global min at $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$K = \{ x_1^2 + 4x_2^2 \leq 1 \text{ and } x_1 + x_2 \geq 1 \}$$
$$= \{ x_1^2 + 4x_2^2 \leq 1 \} \cap \{ x_1 + x_2 \geq 1 \}$$



intersection of 2 convex sets is convex.

isovalues of f level curves

* 11) $x_1 = x_0 - \frac{5}{18}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{5}{18}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} \frac{4}{9} \\ -\frac{1}{9} \end{pmatrix}$

$$x_1 + x_2 = 1 \qquad z_1 = 0$$

$$4x_1 + 2x_2 = 2x_1 + 6x_2$$

$$4x_1 + 2x_2 - z_2 = 0$$

$$2x_1 + 6x_2 - z_2 = 0$$

$$x_1^2 + 4x_2^2 = \frac{4+4}{9} = \frac{8}{9} < 1$$

$$2x_1 = 4x_2$$

$$x_1 = 2x_2$$

$$3x_2 = 1 \qquad x_1 = \frac{2}{3}$$

$$x_2 = \frac{1}{3}$$

$$\inf_{C(x,y,z)=0} f(x,y,z)$$

$f(x,y,z) = x^2 + y^2 + z^2 = \|X\|^2$

$C(x,y,z) = x + y + z - 1$

1) f is coercive and continuous
$\Rightarrow f$ has a local minimum

$\mathcal{D}_a = \{ C(x,y,z) = 0 \}$ is a plane of $\perp$ vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ going through $\begin{pmatrix}1\\0\\0\end{pmatrix}$

the level curves of f are spheres

2) verify qualification of constraint.

$JC(X) = \begin{pmatrix}1 & 1 & 1\end{pmatrix}$ $rk\, JC(X) = 1$ good

3) $\ell(X, \lambda) = f(X) + \lambda(x + y + z - 1)$

$$\nabla_X \ell = 2X + \lambda \begin{pmatrix}1\\1\\1\end{pmatrix} = 0$$

$x = y = z = -\lambda/2$

$-\frac{3\lambda}{2} - 1 = 0$ $\lambda = \frac{-2}{3}$

$x = y = z = \frac{1}{3}$

$$\inf_{X \in K} f(X) = x + y + z \qquad K = \left\{ \begin{array}{l} x^2+y^2+z^2 = 2 \text{ and} \\ (x-1)^2 + y^2 + (z-1)^2 = 1 \end{array} \right\}$$

2) develop the 2nd sphere and substract the 2 equations

$$-2x + 1 - 2z + 1 = -1$$

$$2x + 2z = 3$$ Plane of $\perp$ vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ going through point $\begin{pmatrix} 0 \\ 0 \\ 3/2 \end{pmatrix}$

$$C_3(X) = 2x + 2z - 3$$

1) K is bounded ($\cap$ of 2 spheres) and f continuous therefore X^* exists

3) check the qualification of constraints

$$JC(X) = \begin{pmatrix} 2x & 2y & 2z \\ 2(x-1) & 2y & 2(z-1) \end{pmatrix}$$

$X^* \neq 0_{\mathbb{R}^3}$ because $0 \notin K$

$X^* \neq \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ because $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \notin K$

$$\Rightarrow \operatorname{Rk} JC(K) = 2$$

$$\ell(X,\lambda) = f(X) + \lambda_1 C_1(X) + \lambda_2 C_3(X)$$ simpler

$$= x + y + z + \lambda_1\left(\|X\|^2 - 2\right) + \lambda_2\left(2x + 2z - 3\right)$$

$$\nabla \ell(X,\lambda) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\lambda_1 X + 2\lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Alternative Lagrangian

$$\ell(X,\lambda) = f(X) + \lambda_1 C_1(X) + \lambda_2 C_2(X)$$

$$\nabla \ell(X^*,\lambda^*) = 0 \iff \begin{cases} 1 + 2\lambda_1 x + 2\lambda_2 = 0 \\ 1 + 2\lambda_1 y = 0 \\ 1 + 2\lambda_1 z + 2\lambda_2 = 0 \end{cases}$$

$y = -1/2\lambda_1$ because $\lambda_1 \neq 0$

$(1) + (3) \Rightarrow x = z$

$X^* \in K \qquad 2x + 2z - 3 = 4x - 3 = 0 \qquad x = \frac{3}{4}, \; z = \frac{3}{4}$

$\|X\|^2 = 2 \qquad y^2 = 2 - 2\left(\frac{3}{4}\right)^2 = 2\,\frac{7}{16} = \frac{7}{8}$

$y = \pm\sqrt{\frac{7}{8}}$

1st case: $X^* = \left(\frac{3}{4}, \sqrt{\frac{7}{8}}, \frac{3}{4}\right) \qquad f(X^*) = \frac{6}{4} + \sqrt{\frac{7}{8}} = \frac{3}{2} + \sqrt{\frac{7}{8}}$

2nd case: $X^* = \left(\frac{3}{4}, -\sqrt{\frac{7}{8}}, \frac{3}{4}\right) \qquad f(X^*) = \frac{3}{2} - \sqrt{\frac{7}{8}}$

One minimum and one maximum on a bounded set.