Exemple 2

Alternative method to minimize

$f(x) = 3x_1^2 + 5x_2^2 - 3x_1x_2$

$\mathcal{D}_a : \{x_1 + x_2 = 1\}$

express $x_2 = 1 - x_1$ $\quad g(x_1)$

$c(x) = x_1 + x_2 - 1$.

$c(x) = 0$

$c(x) = \varepsilon > 0$

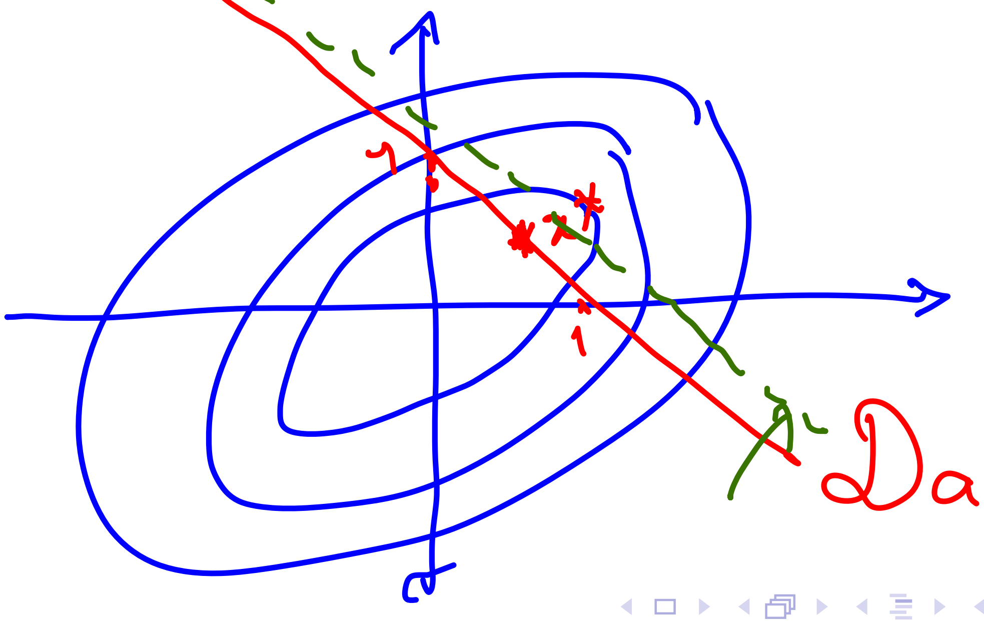
$c(x) = \varepsilon \quad p^*(\varepsilon)$

$y^* = -\dfrac{\partial p(0)}{\partial \varepsilon}$ (slide 245)

linear

$y^* = \dfrac{-51}{22}$

- Solve the system of 3 equations to find $x^\star, y$
- Other method ? $\quad x_1 = \dfrac{13}{22} \quad x_2 = \dfrac{9}{22} \quad p^* = f(x^*)$



Optimality conditions for the optimization with constraints of inequality

$J = \begin{pmatrix} JC^E(x^*) \\ \tilde{J}C^I(x^*) \end{pmatrix}$

$\tilde{J}C^I(x^*)$ = the rows of $JC^I(x^*)$ when $C^I_k(x^*) = 0$

Rank J = maximal (= nb of rows)

Theorem

$x^\star$ is a local minimizer of f verifying the constraints of inequality $c_I(x) \leq 0$ and the constraints of equality $c_E(x^\star) = 0$. If the constraints are qualified, there exists a vector $y^\star \in \mathbb{R}^m$ and a vector $z^\star \in \mathbb{R}^{+p}$ of Lagrange multipliers such as

$$\begin{array}{rrcll} & c^E(x^\star) = 0, c^I(x^\star) & \leq & 0 & \textit{primal feasibility} \\ \forall x \in \mathbb{R}^n & \ell(x^\star, y^\star, z^\star) & \leq & \ell(x, y^\star, z^\star) & \textit{dual feasibility} \\ & z^\star & \geq & 0 & \textit{dual feasibility} \\ \text{for } i=1,\dots,p & c^I_i(x^\star) z^\star_i & = & 0 & \textit{complementary relaxation} \end{array}$$

Comes from $p^* = d^*$

Conditions of complementary relaxation

$p^* = \inf f(x) = f(x^*)$

$g(y^*, z^*) = d^* = \sup g(y,z) = \sup_{y,z} \inf_x f(x) + \langle y, C^E(x)\rangle + \langle z, C^I(x)\rangle$

If $p^* = d^*$ because constraints are qualified

$$p^* = d^* = \inf_x f(x) + \langle y^*, C^E(x)\rangle + \langle z^*, C^I(x)\rangle = g(y^*, z^*)$$

$$f(x^*) \leq f(x^*) + \underbrace{\langle y^*, \underbrace{C^E(x^*)}_{0}\rangle + \langle z^*, C^I(x^*)\rangle}_{=0} \leq f(x^*)$$

then

$$\langle z^*, C^I(x^*)\rangle = 0 \Rightarrow z_j^* C^I(x^*)_j = 0 \ \forall\, j = 1, \ldots, p$$

$\sum_{k=1}^{p} z_k^* C_k^I(x^*) = 0$

with $z_k^* \geq 0$ and $C_k^I(x^*) \leq 0$

First order optimality conditions for the optimization with constraints of inequality

Theorem

Karush-Kuhn-Tucker (KKT) conditions
Let f, c^I and c^E in C^1, and $x^\star$ a local minimizer of f satisfying the inequality constraints $c^I(x) \leq 0$ and equality constraints $c^E(x^\star) = 0$. If the constraints are qualified, there exists $y^\star \in \mathbb{R}^m$ and $z^\star \in \mathbb{R}^{+p}$ Lagrange multipliers such that

$$
\begin{array}{rcll}
c^E(x^\star) = 0, c^I(x^\star) & \leq & 0 & \textit{primal feasibility} \quad \mathcal{D}_a \\
g(x^\star) + {A^E}^T(x^\star)y^\star + {A^I}^T(x^\star)z^\star & = & 0 & \textit{dual feasibility} \quad \nabla_x \ell(x^\star, y^\star, z^\star) = 0 \\
z^\star & \geq & 0 & \textit{dual feasibility} \\
\text{OK} \rightarrow \quad \forall i = 1, \ldots, m \quad c_i^I(x^\star)z_i^\star & = & 0 & \textit{complementary relaxation}
\end{array}
$$

KKT Conditions deduced from Lagrange multipliers theorem

Replace the original inegality constrained problem by

inf $x \in \mathbb{R}^n, t \in \mathbb{R}^p F(x,t)$ with

$F(\underbrace{x,t}) = f(x)$ both primal variable

and equality constraints

$c_i^E(x) = 0$ pour $i = 1, \ldots, m$

$c_j^I(x) + t_j^2 = 0$ pour $j = 1, \ldots, p$.

$\inf F(x)$
$c^E(x) = 0$
$c^I(x) \leq 0$
original

The lagrangian of the modified problem is

$$L(x,t,y,z) = F(x,t) + \sum_{i=1}^{m} y_i c_i^E(x) + \sum_{j=1}^{p} z_j (c_j^I(x) + t_j^2)$$

Lagrange multipliers theorem provides

$$\nabla_{x,t} F(x,t) + \sum_{i=1}^{m} y_i \nabla_{x,t} c_i^E(x) + \sum_{j=1}^{p} z_j \nabla_{x,t} (c_j^I(x) + t_j^2) \quad = \quad 0$$

KKT proof... $\mathcal{L}(x,t,y,z) = F(x,t) + \sum y_i c^E(x) + \sum z_j (c_j^I(x) + t_j^2)$

$\partial / \partial x_i$

n equations

$$
\begin{aligned}
\nabla_x f(x) + \sum_{i=1}^{m} y_i \nabla_x c_i^E(x) + \sum_{j=1}^{p} z_j \nabla_x (c_j^I(x) &= 0 \\
2 z_j t_j &= 0, \quad j = 1, \ldots, p
\end{aligned}
$$

Relaxation $t_j = 0 \Leftrightarrow c_j^I(x) = 0 \;\; (\Longleftarrow)$

Condition $z_j \geq 0$ To find this condition, we apply the 2nd order optimality condition on the Lagrangian of $F(x, t)$:

$$
H_{x,t} L(x,t,y,z) = \begin{pmatrix} H_x \ell(x,y,z) & 0 \\ 0 & \begin{pmatrix} 2z_1 & 0 & \ddots & 0 \\ 0 & 2z_2 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & 0 & 2z_p \end{pmatrix} \end{pmatrix}
$$

$H_{x,t} \mathcal{L} \geq 0$

$\Rightarrow z_j \geq 0$

$j = 1, \ldots, p$

Example of KKT application

$\ell(x, z) = \|x\|^2 + z(x_1 + x_2 - 1)$

We look at the quadratic minimization problem

$$\inf_{x_1+x_2-1\leq 0} x_1^2 + x_2^2.$$

Trivial solution: $(0,0)$ checks the inequality constraint therefore the constraint is inactive in $x^\star$, the solution of the problem is the solution of the unconstrained problem, i.e. $(0,0)$.

KKT check : We seek $(x^\star, z^\star)$ with $\boxed{z^\star \geq 0}$ s. t.

$$\nabla_x \ell(x^\star, z^\star) = \begin{pmatrix} 2x_1^\star \\ 2x_2^\star \end{pmatrix} + z^\star \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\boxed{z^\star(x_1^\star + x_2^\star - 1) = 0.}$$

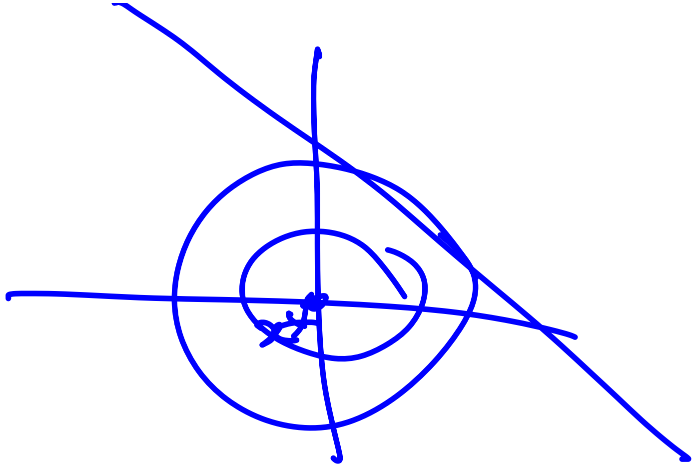
Relaxation

Example of KKT application

$$2x^\star + z^\star \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$(x_1^\star + x_2^\star - 1) z^\star = 0$$

$$(-z - 1) z = 0$$



From the two first equalities $x_1^\star = x_2^\star = -z^\star/2$
replace in the third one leads to
either $z^\star = 0$ then $x_1^\star = x_2^\star = 0$
either $x_1^\star = x_2^\star = 1/2$ then $z^\star = -1 < 0$ impossible.

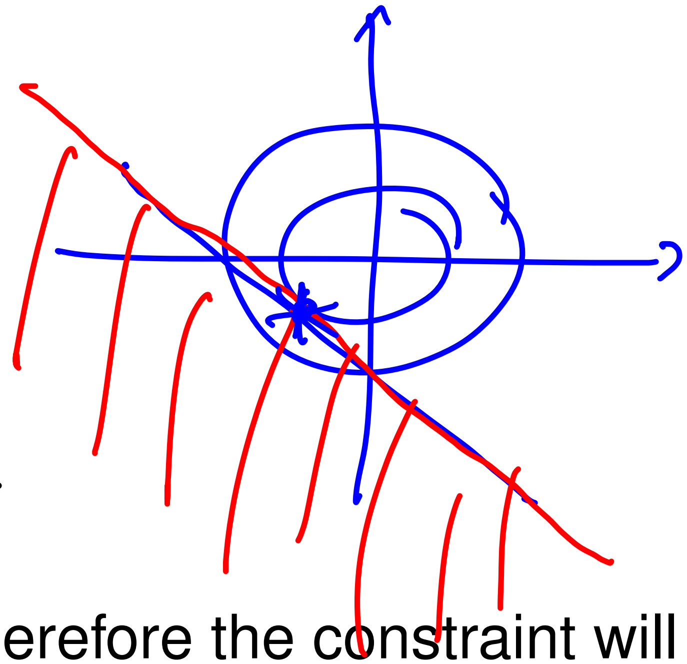
Modified example

$2x + 3\begin{pmatrix}1\\1\end{pmatrix} = 0 \Rightarrow x_1 = x_2 = -3/2$

$3(x_1 + x_2 + 1) = 0$

$3(-3+1) = 0$

$x^* = \left(\frac{-1}{2}, \frac{-1}{2}\right)$ with $z^* = 1$



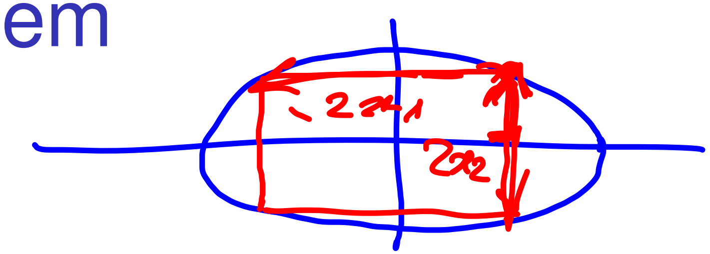
$$\inf_{x_1+x_2+1\leq 0} x_1^2 + x_2^2.$$

$(0,0)$ does not satisfy the constraint therefore the constraint will be active in $x^\star$.
The third KKT condition is now $z^\star(x_1^\star + x_2^\star + 1) = 0 \Rightarrow$
either $z^\star = 0$ then $x_1^\star = x_2^\star = 0$, does not satisfy the constraint
either $x_1^\star = x_2^\star = -1/2$ then $z^\star = 1$, correct solution.
Verification : change of variable $x_2 = -1 - x_1$ in the function :
$\inf_{x_1} x_1^2 + (1 + x_1)^2$ is attained at $x_1 = -1/2$.

Exemple Kepler's problem

$V(x) = x_1 x_2 x_3$

$f(x) = -x_1 x_2 x_3$



$J_C(x) = 2\left(\frac{x_1}{a_1^2}, \frac{x_2}{a_2^2}, \frac{x_3}{a_3^2}\right) \quad x^* \neq 0 \Rightarrow \mathrm{rk}\, J_C(x^*) = 1$

Find the parallelepiped of maximum volume inscribed in the ellipsoid

$\mathcal{D}_a = \{ x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 \leq 1 \}$

$$\mathcal{E} = \{x \in \mathbb{R}^3, {x_1}^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 = 1\}$$

$c(x) = x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 - 1$

Write the problem as a canonical optimisation problem

$\ell(x,y) = -x_1 x_2 x_3 + y\left(x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 - 1\right)$

$$\nabla_x \ell(x,y) = \begin{pmatrix} -x_2 x_3 \\ -x_1 x_3 \\ -x_1 x_2 \end{pmatrix} + 2y \begin{pmatrix} x_1/a_1^2 \\ x_2/a_2^2 \\ x_3/a_3^2 \end{pmatrix} = 0$$

Exemple Kepler's problem

Find the parallelepiped of maximum volume inscribed in the ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^3, {x_1}^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 = 1\}$$

Write the problem as a canonical optimisation problem

$$\inf_{\substack{{x_1}^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 = 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0}} f(x), \quad f(x) = -\prod_{i=1}^{3} x_i$$

these constraints are necessarily inactive at x^* because otherwise $V(x^*) = 0$

- Can we apply KKT theorem?
- Which constraints are active ?
- Lagrangian ?

Exemple Kepler's problem

$x_1 = 0$ or $x_2 = 0$ or $x_3 = 0 \Rightarrow f(x) = 0$! inequality constraints are inactive

$$\ell(x, y) = -\prod_{i=1}^{3} x_i + y({x_1}^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 - 1)$$

Gradient

Exemple Kepler's problem

$$
\begin{aligned}
\Big(-x_2 x_3 + 2 y x_1 / a_1^2 &= 0 \Big) \times x_1 \\
\Big(-x_1 x_3 + 2 y x_2 / a_2^2 &= 0 \Big) \times x_2 \\
\Big(-x_2 x_1 + 2 y x_3 / a_3^2 &= 0 \Big) \times x_3
\end{aligned}
$$

then sum the 3 equations:

then find x^*