Outline

Numerical methods for optimisation

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We consider :
a function $v : \mathbb{R}^n \to \mathbb{R}^p$ twice differentiable on $\mathbb{R}^n$, with Jacobian matrix $J(x)$,
a positive definite symmetric matrix $A \in \mathbb{R}^{p \times p}$,
and the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \langle Av(x), v(x) \rangle.$$

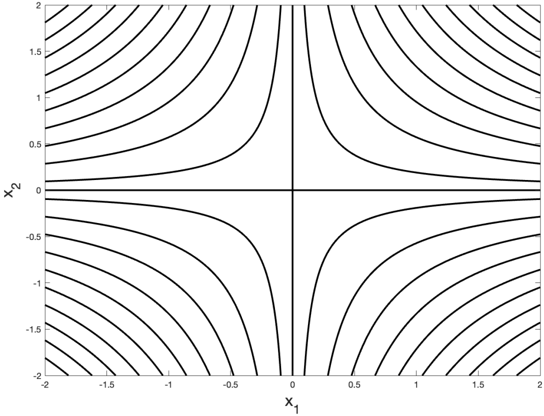
The gradient of $f(x)$ is

A) $\nabla f(x) = (J^T + J)Av(x)$

B) $\nabla f(x) = 2J^T Av(x)$

C) $\nabla f(x) = J(A + A^T)v(x)$

Answer question 285 on https://toreply.univ-lille.fr



The graph represents the isovalues of

A) $f(x) = x_1 + x_2$

B) $f(x) = x_1 x_2$

C) $f(x) = x_1^2 - x_2^2$

Answer question 131 on https://toreply.univ-lille.fr

$$f(x) = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}(x_2 + 1)^2.$$

The hessian matrix of f is

A) $Hf(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$

B) $Hf(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$

C) $Hf(x) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$

Answer question 221 on https://toreply.univ-lille.fr

Epigraph

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2. Defining the epigraph of $f : \mathbb{R}^n \to \mathbb{R}$

$$\textit{epi } f = \{(x, t) \in \mathbb{R}^{n+1}, x \in \text{dom } f,\ f(x) \leq t\}$$

f convex if and only if epi f is convex

Convexity of a differentiable function

Property : Let f be a function defined on a real-valued differentiable convex $C \subset \mathbb{R}^n$. The function f is

1. convex if and only if
   $\forall (x, y) \in C^2, \langle \nabla f(x), y - x \rangle \leq f(y) - f(x)$.

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2. convex if and only if
   $\forall (x, y) \in C^2, \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$.
3. strictly convex if and only if
   $\forall (x, y) \in C^2, x \neq y, \langle \nabla f(x), y - x \rangle < f(y) - f(x)$.

Strong convexity

Definition : Let $K \subset V$ be a convex. A function $f : K \to \mathbb{R}$ is said to be **strongly convex** or **uniformly convex** or α**-convex** or α**-elliptical** if there exists $\alpha > 0$ such that

$$\forall (x, y) \in K^2, \forall \lambda \in [0, 1],$$
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - \frac{\alpha}{2}\lambda(1-\lambda)\|xy\|^2.$$

Characterization of strong convexity

1. If $f : V \to \mathbb{R}$ is continuous, the following properties are equivalent
   1.1 f is α-elliptical
   1.2 For all $(x, y) \in V^2$, $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \frac{\alpha}{8}\|xy\|^2$
2. If $f : V \to \mathbb{R}$ is differentiable, the following properties are equivalent
   2.1 f is α-elliptical
   2.2 For all $(x, y) \in V^2$, $f(y) \geq f(x) + \langle \nabla f(x), yx \rangle + \frac{\alpha}{2}\|xy\|^2$
   2.3 For all $(x, y) \in V^2$, $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \alpha\|x - y\|^2$

Convexity of a twice differentiable function

$f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable on dom f convex, Hessian matrix $Hf(x) \in S^n$ (or sometimes $\nabla^2 f(x)$)

- f is convex iff $Hf(x) \in S^n_+ \quad \forall x \in \text{dom} f$ (positive semi-definite)
- if $Hf(x) \in \ S^n_{++} \quad \forall x \in \text{dom} f$ (positive definite) then f is strictly convex

Characterization of the strong convexity of a twice differentiable function

If $f : V \to \mathbb{R}$ is twice differentiable, the following properties are equivalent

1. f is α-elliptical
2. For all $(x, h) \in V^2$, $\langle \mathrm{H}\, f(x)h, h\rangle \geq \alpha\|h\|^2$

Examples

Quadratic function

$$f(x) = \frac{1}{2}x^t P x + q^t x + r$$

with $P \in S^n$
Least squares $f(x) = \|Ax - b\|_2^2$
Quadratic on linear $f(x, y) = x^2/y$ on $\mathbb{R} \times \mathbb{R}^{+\star}$

Practical methods to show that a function is convex

- Check definition (Jensen)
- Check definition on lines
- For twice differentiable functions, show that the Hessian is positive
- Show that the function is obtained from convex functions by operations which preserve convexity

Linear operations

- Multiplication of f by a positive constant
- Sum (finite or infinite, integral)
- Composition with an affine function

Examples

- Logarithmic barriers for affine inequalities

$$
\begin{aligned}
f(x) &= -\sum_{i=1}^{m} \log(b_i - a_i^t x), \\
\text{dom } f &= \{x, a_i^t x < b_i,\ i = 1,\ ldots, m\}
\end{aligned}
$$

- Norm of an affine function